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" *To the solid ground*
Of Nature trusts the mind which builds for aye."—WORDSWORTH.

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LINEAR DIFFERENTIAL EQUATIONS.

Theory of Differential Equations. By A. R. Forsyth, Sc.D., LL.D., F.R.S. Part iii. Ordinary Linear Equations. Vol. iv. Pp. xvi + 534. (Cambridge: The University Press, 1902.)

IN this volume Prof. Forsyth deals with a part of his subject which, for many reasons, is full of interest. Ordinary linear differential equations concern the physicist, on the one hand, by their occurrence in the analysis required for many of his most important problems; on the other, they offer the pure mathematician an attractive field of research which appears to be almost inexhaustible.

Thanks to the contributions of a host of analysts, the theory of linear equations has now reached a high stage of development, and, as in other like cases, it is extremely interesting to see how different parts of it, which at first seemed isolated, are being gradually brought into organic connection. One of the first great steps in this direction was made by Gauss in his memoir on the hypergeometric series; this is another example of the extraordinary and almost uncanny way in which Gauss transformed and generalised every subject that he touched. It is as if his predecessors had been hewing stones for him to fit together into the lower courses of a stately building which he left for others to complete. And worthy successors have not been wanting, of whom, perhaps, Riemann is as yet the chief. For his brief memoir on the P-function marks an epoch by introducing several new notions of the very highest importance—the indices associated with the critical points, the analytical continuation of a branch of the function which satisfies the equation and the group of linear substitutions generated by describing cycles including critical points.

The real significance of Riemann's paper became fully evident only after the appearance of the celebrated memoir of Fuchs. It is, of course, impossible to say how Fuchs arrived at his discoveries; very likely he could not have explained his induction completely himself. In the introduction he refers to Briot and Bouquet as well as to Riemann, and acknowledges his obligations to Weierstrass. Fuchs deals with an equation of quite general order, the coefficients being functions of x with a limited number of singularities. He shows that in the neighbourhood of each critical point a there is a solution of the form $(x-a)^k\phi$, where ϕ is a one-valued analytical function and k is a constant determined by an equation which can be constructed from the differential equation itself. He also shows how the simplest independent solutions group themselves according to the multiplicities of the roots of the indicial equation.

The importance of these expansions near the critical points is that, besides giving us information about the analytical properties of the function defined by the differential equation, they enable us to investigate the group of substitutions associated with it. Suppose, for instance, we have an equation of the second order, and that in the neighbourhood of a there are two solutions of the form $(x-a)^h\phi$ and $(x-a)^k\psi$; if the independent variable starts near a and describes a small circuit round it, the solutions, by continuous variation, are multiplied by $e^{2\pi hi}$ and $e^{2\pi ki}$ respectively; thus with these solutions we have a substitution of the form $y'_1 = sy_1$, $y'_2 = ty_2$, where s, t are constants. When the indicial equation for a has multiple roots, the associated substitution is less simple, but can always be determined. If we start from *any* ordinary point with a set of independent solutions, then by Weierstrass's principle of continuation we can (in theory at least) follow up their values as x approaches a critical point a, then find the substitution which takes place as x goes round a, and finally bring back x to its original position. The effect of any closed circuit can thus be determined; and we have, on the whole, a group of linear substitutions, with generators corresponding to the critical points.

The singularities of an integral are determined by the coefficients of the differential equation; they may be poles or they may be essential singularities. One of the most remarkable things in Fuchs's paper is the determination of the form which a differential equation must have if all its integrals are regular in the neighbourhood

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of each critical point; that is to say, if near any critical point a each integral can be put into the form

$$y = (x-a)^k \left\{ \phi_0 + \phi_1 \log(x-a) + \ldots + \phi_m [\log(x-a)]^m \right\}$$

where $\phi_0, \phi_1, \ldots \phi_m$ are one-valued functions not infinite at a. These equations are called by Prof. Forsyth "equations of Fuchsian type." The equation of the hypergeometric series is of this type, and is remarkable as being the only one, of order higher than the first, which is completely determined when the positions of the critical points and the indices associated with them are assigned.

An equation of Fuchsian type may have one or more algebraic integrals. If all the integrals are algebraic, the group of the equation must be finite; so here we have a most unexpected concurrence of two apparently disconnected theories. A very interesting problem is that of determining linear equations the groups of which are isomorphic with known finite groups; another is that of finding out whether a given equation has any algebraic integrals.

All the foregoing theory is discussed and illustrated by Prof. Forsyth in a very attractive and lucid manner; thus chapter i. deals with the existence of a synectic integral near an ordinary point and sets of independent integrals; chapter ii. with the expansions near a critical point and with Hamburger's method of grouping them; chapter iii. with regular integrals; chapter iv. with equations of Fuchsian type; and chapter v. with equations of the second and third orders possessing algebraic integrals. Illustrations are supplied by the familiar equations of mathematical physics, by the equation of the elliptic quarter-period, and by that of the hypergeometric series. It is delightful to see how the discussion of these equations is illuminated by the general theory.

After a chapter on equations with only some of their integrals regular, we come to the consideration of integrals with essential singularities. The most familiar example of a function with an essential singularity at a finite place is exp (x^{-1}), which is the integral of $x^2y' + y = 0$; and it is easy to see that if P is any polynomial in x^{-1}, the expression expP has an essential singularity and satisfies a linear equation of the first order.

Suppose now that we find that a given equation has an integral with an essential singularity at the origin; it may be possible to express it in the form $\exp P . x^\rho \psi(x)$, where ρ is constant and $\psi(x)$ holomorphic. Such an integral has been called "normal"; the discussion of these integrals, and others obtained by putting $x^{1/k}$ for x, is given in chapter vii., which contains important results due to Thomé, Hamburger, Poincaré and others. There is also a brief account of "double-loop integrals" after Jordan and Pochhammer, and of Poincaré's theory of asymptotic integrals.

In his paper on the motion of the moon, Hill was led to the solution of a linear equation by a method involving the use of infinite determinants. In chapter viii. Prof. Forsyth discusses this method in some detail, after giving a preliminary account of infinite determinants and their properties. The subject of this chapter is not very attractive in itself, but on account of its practical importance has naturally attracted a good deal of attention.

Chapter ix. deals with equations with uniform periodic coefficients, and gives an account of this part of the subject which ought to encourage young mathematicians to read the original sources and experiment on their own account. It is, of course, the equations with doubly periodic coefficients that are most interesting. Thanks principally to Hermite, Halphen and Picard, some extremely beautiful results have been already obtained in this field, and there can be no doubt that others are awaiting discovery.

The last chapter of this volume, on equations with algebraic coefficients, must have been very difficult to write, and appeals mainly to the specialist. Its principal topic is Poincaré's celebrated theorem that the integrals of any linear equation with algebraic coefficients can be expressed by means of Fuchsian and Zetafuchsian functions. As Prof. Forsyth justly remarks, we cannot hope to make practical use of Poincaré's theorem until the analysis of automorphic functions has reached a higher state of development. To this end the treatise by Klein and Fricke, now in course of publication, will doubtless contribute largely.

In conclusion, it may be well to remark that this volume is in great measure independent of its predecessors, and that a great part of it will be quite intelligible to junior mathematicians provided that they know the elements of the theory of a complex variable. To them, therefore, as well as to their seniors, this book may be heartily commended. G. B. M.

SCIENTIFIC PSYCHOLOGY.

Grundzüge der physiologischen Psychologie. Von Wilhelm Wundt. Fünfte völlig umgearbeitete Auflage. Erster Band. Pp. xv + 553. (Leipzig: W. Engelmann, 1902.) Price 10*s*. net.

THIS volume of 553 pages is the first of the three volumes in which the fifth edition of Prof. Wundt's great work is to appear. The rapid increase in size of the work in each of the successive editions is thus maintained in the present one, and, as in the case of the previous editions, has been necessitated by the rapidity of the growth of the youngest of the natural sciences, experimental or, as Prof. Wundt prefers to call it, physiological psychology. And even the increase in bulk of this book does not by any means fully express the rate of growth of the science, a growth towards which this country has contributed so lamentably little. For the book is primarily a record of the work and the views of the author and of his pupils in the great Leipzig school. Nevertheless, Prof. Wundt has found it necessary to rewrite almost the whole of the book, so that, as he tells us, it must be regarded as almost a new one.

The greater part of this first volume is concerned with matters not strictly psychological, but rather with those studies which form an essential part of the equipment of the psychologist, namely, the fine and coarse anatomy, the embryology and the physiology of nervous tissues, both special and comparative. It is, perhaps, open to question whether it is wise to attempt to treat so vast a range of subjects in the scope of a single volume. For

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